Dodecahedral topology fails to explain quadrupole-octupole alignment

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Abstract

The CMB quadrupole and octupole, as well as being weaker than expected, align suspiciously well with each other. Non-trivial spatial topology can explain the weakness. Might it also explain the alignment? The answer, at least in the case of the Poincaré dodecahedral space, is a resounding no.

1 Introduction

Soon after the release of the first-year WMAP data [1], Tegmark *et al.* [2] noticed that the CMB quadrupole and octupole aligned with each other unusually well, at roughly the 98% level. Multipole vectors – discovered by Maxwell [3] in the 19^{th} century, widely forgotten, then reintroduced by Copi *et al.* [4] – provide a useful tool for analyzing the alignment in greater detail. While exact confidence levels vary depending on what one measures, all researchers agree that the quadrupole-octupole alignment is unusual at roughly the 99% level or better [5, 6, 7]. The combination of the 1-in-100 alignment with the 1-in-600 overall weakness of the low-ℓ modes motivates one to seek a physical explanation.

Non-trivial spatial topology can explain the weakness of the low-ℓ modes. Might it also explain the quadrupole-octupole alignment? The present paper simulates the CMB in a Poincaré dodecahedral space [8, 9] and checks the quadrupole-octupole alignment. Absolutely no correlation is found.

2 Simulating the space

We use the late Jesper Gundermann's simulation [10] of the CMB in the Poincaré dodecahedral space, with modes through $k_{\max} = 102$. This simplified simulation, while neglecting the Doppler contribution and the sound speed, nevertheless produces a low-ℓ power spectrum essentially identical to the spectra produced by more refined simulations. Thus we may be quite confident that if the dodecahedral topology imposed a nontrivial quadrupole-octupole alignment, this simulation would capture it. As we will see in Section 3, however, absolutely no such correlation is found. Even if one were to add a Doppler term and sound speed to the simulation, the distribution in Figure 1 would change by at most a tiny amount, not nearly enough to introduce a nontrivial quadrupole-octupole correlation.

3 Measuring the alignment

For each simulated CMB sky, we use the polynomial method [6] to compute the two quadrupole vectors $\{\mathbf{u}_{2,1}, \mathbf{u}_{2,2}\}$ and the three octupole vectors $\{\mathbf{u}_{3,1}, \mathbf{u}_{3,2}, \mathbf{u}_{3,3}\}$. Following [5], we take the cross product $\mathbf{w}_2 = \mathbf{u}_{2,1} \times \mathbf{u}_{2,2}$, which we normalize to obtain a unit vector $\mathbf{n}_2 = \mathbf{w}_2/|\mathbf{w}_2|$ orthogonal to the plane of the quadrupole. Similarly, we take the cross product of each of the three possible pairs of octupole vectors

$$\begin{aligned} \mathbf{w}_{3,1} &= \mathbf{u}_{3,2} \times \mathbf{u}_{3,3} \\ \mathbf{w}_{3,2} &= \mathbf{u}_{3,3} \times \mathbf{u}_{3,1} \\ \mathbf{w}_{3,3} &= \mathbf{u}_{3,1} \times \mathbf{u}_{3,2} \end{aligned} \tag{1}$$

which we normalize to obtain unit vectors $\mathbf{n}_{3,i} = \frac{\mathbf{w}_{3,i}}{|\mathbf{w}_{3,i}|}$ orthogonal to each of the three octupole planes. The three dot products $D_i = |\mathbf{n}_2 \cdot \mathbf{n}_{3,i}|$ then measure the extent to which the quadrupole plane does or does not align with each of the three octupole planes.

In a simply connected universe one expects no correlation between the quadrupole vector $\mathbf{n}_2$ and each octupole vector $\mathbf{n}_{3,i}$. As $\mathbf{n}_2$ and $\mathbf{n}_{3,i}$ (for some fixed i) wander randomly over the 2-sphere, their dot product follows a flat distribution on the interval $[-1, +1]$ (this is a consequence of the wonderful fact that radial projection of a sphere onto a circumscribed cylinder via $(x, y, z) \mapsto (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, z)$ preserves area). Hence each $D_i = |\mathbf{n}_2 \cdot \mathbf{n}_{3,i}|$,

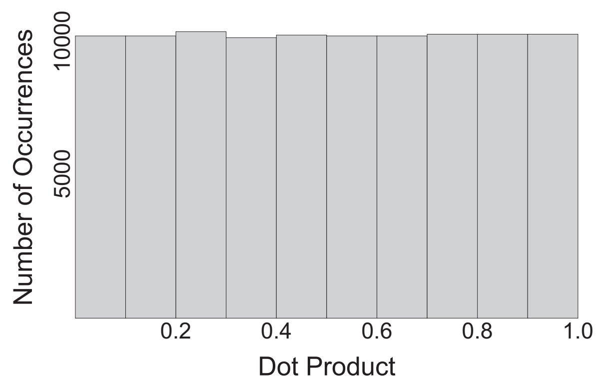

Figure 1: 100000 simulations of the Poincaré dodecahedral space find the distribution of the quadrupole-octupole dot product $|\mathbf{n}_2 \cdot \mathbf{n}_{3,i}|$ to be completely flat, just as in a simply connected universe.

being the absolute value of the dot product, follows a flat distribution on $[0, 1]$.

In the real universe the quadrupole aligns surprisingly well with the octupole, giving dot products $\{D_1, D_2, D_3\} = \{0.84, 0.87, 0.95\}$ for the DQ-corrected Tegmark (DQT) cleaning [2] of the first-year WMAP data or $\{0.85, 0.87, 0.93\}$ for the Lagrange Internal Linear Combination (LILC) cleaning [11] of the same data.

The question of whether a multiconnected spatial topology might explain the observed quadrupole-octupole alignment may be rephrased more precisely as: Does a given topology predict a flat distribution for each D_i or does it predict a distribution skewed towards the high end? For the Poincaré dodecahedral space, our simulations (recall Section 2) yield a flat distribution (Figure 1), implying that the dodecahedral topology does nothing to explain the quadrupole-octupole alignment.

To be fully rigorous we should point out that even though the individual dot products D_i follow the same flat distribution in the dodecahedral topology that they do in the simply connected model, it's nevertheless conceivable that their sum $D_1 + D_2 + D_3$ might follow a slightly different distribution in the two cases, depending on the internal correlations among the three D_i in the dodecahedral case. In practice, however, our simulations find the observed sum to be unusual at roughly the 99% level regardless of whether we compare to the dodecahedral topology or a simply connected space.

4 Conclusion

The Poincaré dodecahedral space topology, while explaining the weakness of the low-ℓ modes, completely fails to explain the quadrupole-octupole alignment. While this negative result leaves one feeling less optimistic, good scientific practice demands that one analyze a few other plausible topologies before reaching any firm conclusion about whether topology might play a role.

One must also keep an open mind about what observations may or may not be due to random chance alone. The quadrupole-octupole alignment might be due to chance, while the weakness of the low-ℓ modes has a physical explanation. Or perhaps exactly the reverse is true. At this point the mystery remains open.

Acknowledgments

I dedicate this article to my friend and collaborator Jesper Gundermann, whose untimely death on 10 June 2006 saddened all who knew him. His energetic enthusiasm and deep love of science brought joy to those of us lucky enough to work with him.

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